# CS 188: Artificial Intelligence

Markov Decision Processes (MDPs)

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Some slides adapted from Dan Klein

# Outline

- Markov Decision Processes (MDPs)
  - Formalism
  - Value iteration
    - In essence a graph search version of expectimax, but
      - there are rewards in every step (rather than a utility just in the terminal node)
      - ran bottom-up (rather than recursively)
      - can handle infinite duration games
  - Policy Evaluation and Policy Iteration

# Non-Deterministic Search

How do you plan when your actions might fail?

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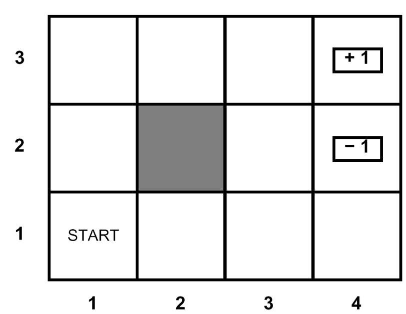
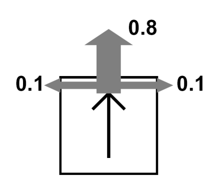
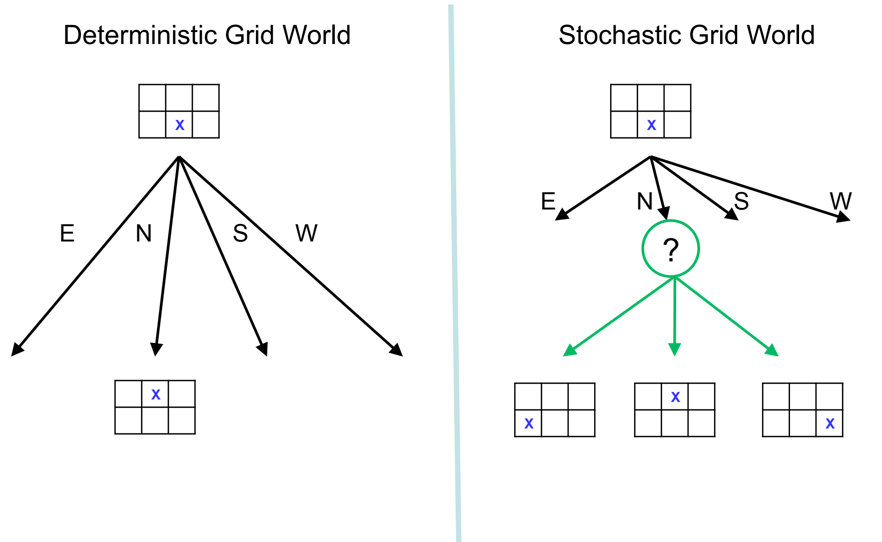
# Grid World

- The agent lives in a grid
- Walls block the agent's path
- The agent's actions do not always go as planned:
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Small "living" reward each step (can be negative)
- Big rewards come at the end
- Goal: maximize sum of rewards



# Grid Futures

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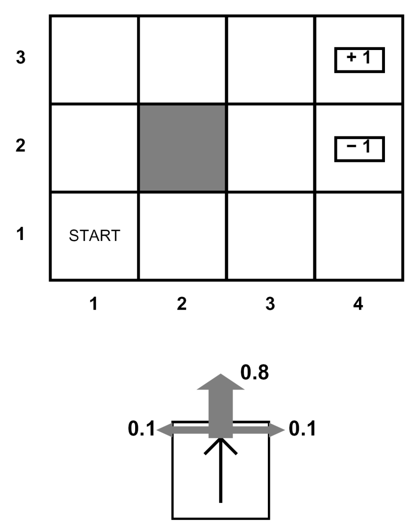
# Markov Decision Processes

- An MDP is defined by:
  - A set of states s ∈ S
  - A set of actions a ∈ A
  - A transition function T(s,a,s')
    - Prob that a from s leads to s'
    - i.e., P(s' | s,a)
    - Also called the model
  - A reward function R(s, a, s')
    - Sometimes just R(s) or R(s')
  - A start state (or distribution)
  - Maybe a terminal state
- MDPs are a family of non-deterministic search problems
  - One way to solve them is with expectimax search – but we'll have a new tool soon

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# What is Markov about MDPs?

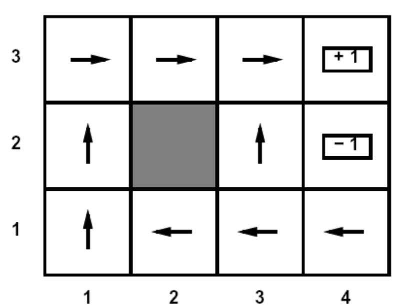
- Andrey Markov (1856-1922)
- "Markov" generally means that given the present state, the future and the past are independent
- For Markov decision processes, "Markov" means:

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots S_0 = s_0)$$
$$=$$
$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

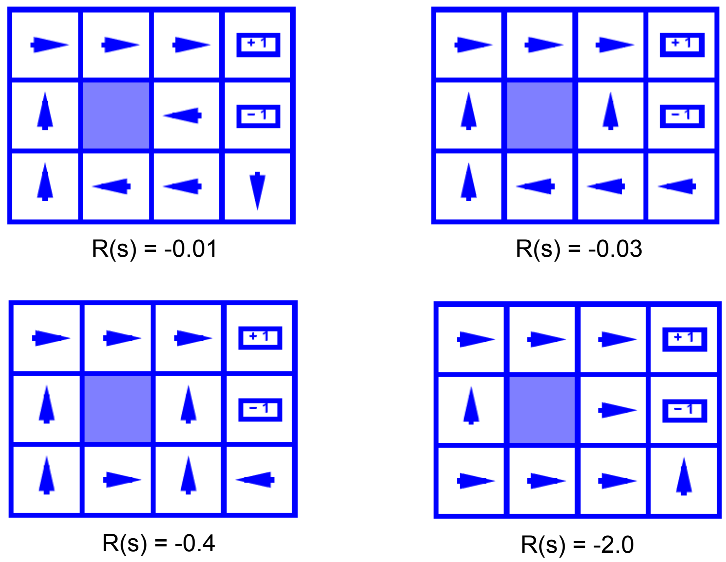
# Solving MDPs

- In deterministic single-agent search problems, want an optimal plan, or sequence of actions, from start to a goal
- In an MDP, we want an optimal policy $\pi^*$: S → A
  - A policy $\pi$ gives an action for each state
  - An optimal policy maximizes expected utility if followed
  - Defines a reflex agent

Optimal policy when R (s, a, s') = -0.03 for all non-terminals s



# Example Optimal Policies

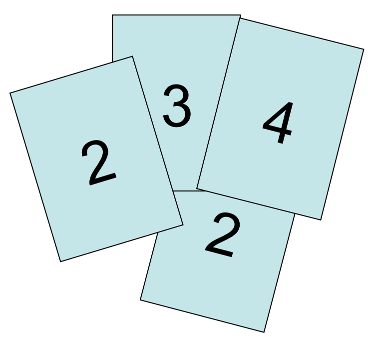





# Example: High-Low

- Three card types: 2, 3, 4
- Infinite deck, twice as many 2's
- Start with 3 showing
- After each card, you say "high" or "low"
- New card is flipped
- If you're right, you win the points shown on the new card
- Ties are no-ops
- If you're wrong, game ends

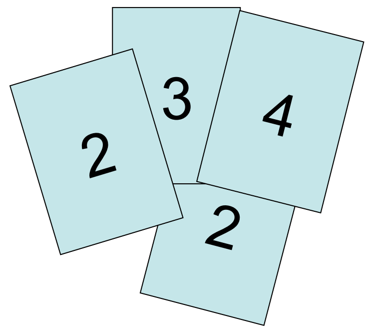
- Differences from expectimax:
  - #1: get rewards as you go --- could modify to pass the sum up
  - #2: you might play forever! --- would need to prune those, we'll see a better way



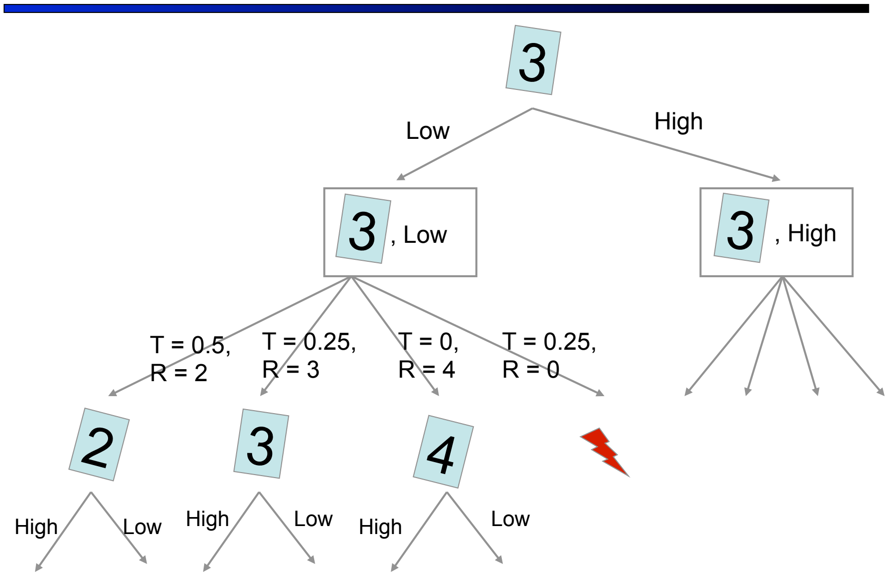

*You can patch expectimax to deal with #1 exactly, but not #2…*

# High-Low as an MDP

- States: 2, 3, 4, done
- Actions: High, Low
- Model: T(s, a, s'):
  - P(s'=4 | 4, Low) = 1/4
  - P(s'=3 | 4, Low) = 1/4
  - P(s'=2 | 4, Low) = 1/2
  - P(s'=done | 4, Low) = 0
  - P(s'=4 | 4, High) = 1/4
  - P(s'=3 | 4, High) = 0
  - P(s'=2 | 4, High) = 0
  - P(s'=done | 4, High) = 3/4
  - …
- Rewards: R(s, a, s'):
  - Number shown on s' if s ≠ s'
  - 0 otherwise
- Start: 3




# Example: High-Low

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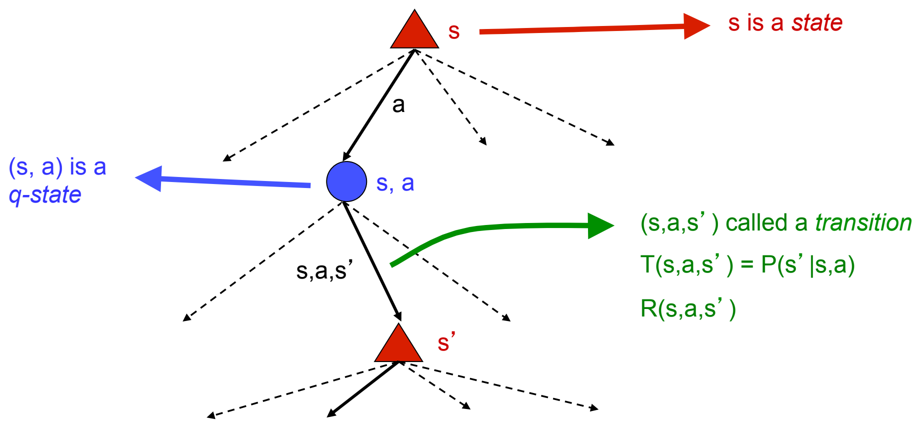
# MDP Search Trees

- Each MDP state gives an expectimax-like search tree

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# Utilities of Sequences

- What utility does a sequence of rewards have?
- Formally, we generally assume stationary preferences:

$$[r, r_0, r_1, r_2, \ldots] \succ [r, r'_0, r'_1, r'_2, \ldots]$$
$$\Leftrightarrow$$
$$[r_0, r_1, r_2, \ldots] \succ [r'_0, r'_1, r'_2, \ldots]$$

- Theorem: only two ways to define stationary utilities
  - Additive utility:

$$U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$$

  - Discounted utility:

$$U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$$

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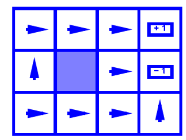
# Infinite Utilities?!

- Problem: infinite state sequences have infinite rewards
- Solutions:



  - Finite horizon:
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies ($\pi$ depends on time left)
  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "done" for High-Low)
  - Discounting: for $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1-\gamma)$$

    - Smaller $\gamma$ means smaller "horizon" – shorter term focus

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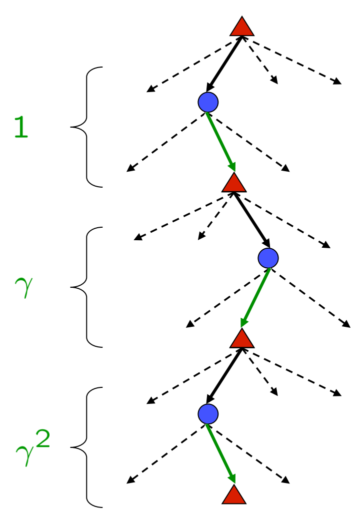
# Discounting

- Typically discount rewards by $\gamma < 1$ each time step
  - Sooner rewards have higher utility than later rewards
  - Also helps the algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])

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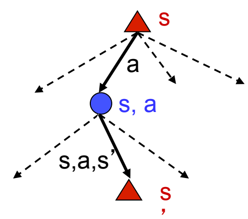
# Recap: Defining MDPs

- Markov decision processes:
  - States S
  - Start state $s_0$
  - Actions A
  - Transitions P(s' |s,a) (or T(s,a,s' ))
  - Rewards R(s,a,s' ) (and discount $\gamma$)




- MDP quantities so far:
  - Policy = Choice of action for each state
  - Utility (or return) = sum of discounted rewards

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# Our Status

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  - Value iteration
    - In essence a graph search version of expectimax, but
      - there are rewards in every step (rather than a utility just in the terminal node)
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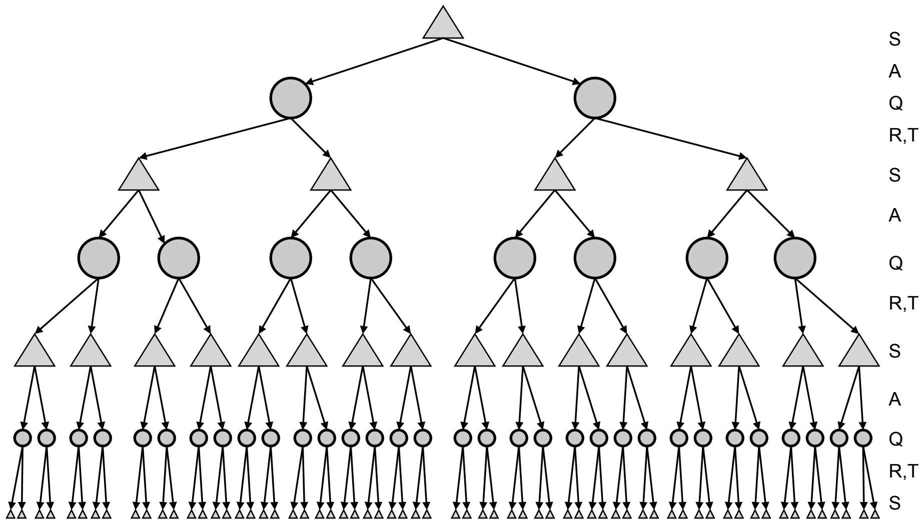
## Expectimax for an MDP

Example MDP used for illustration has two states, S = {A, B}, and two actions, A = {1, 2}

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## Expectimax for an MDP

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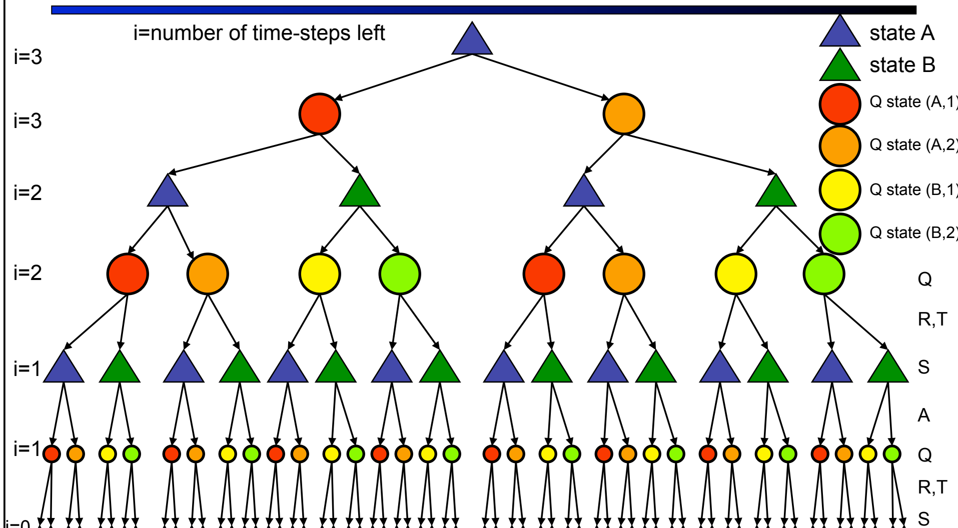


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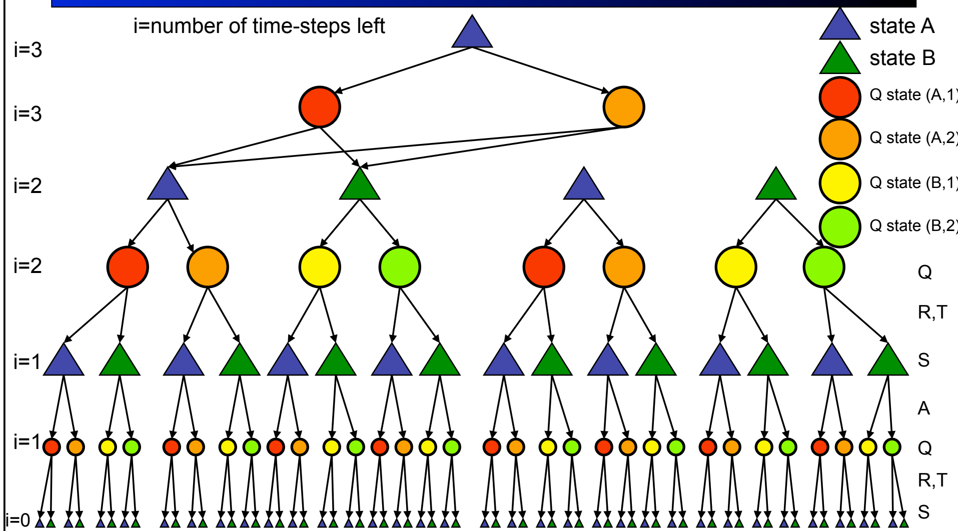


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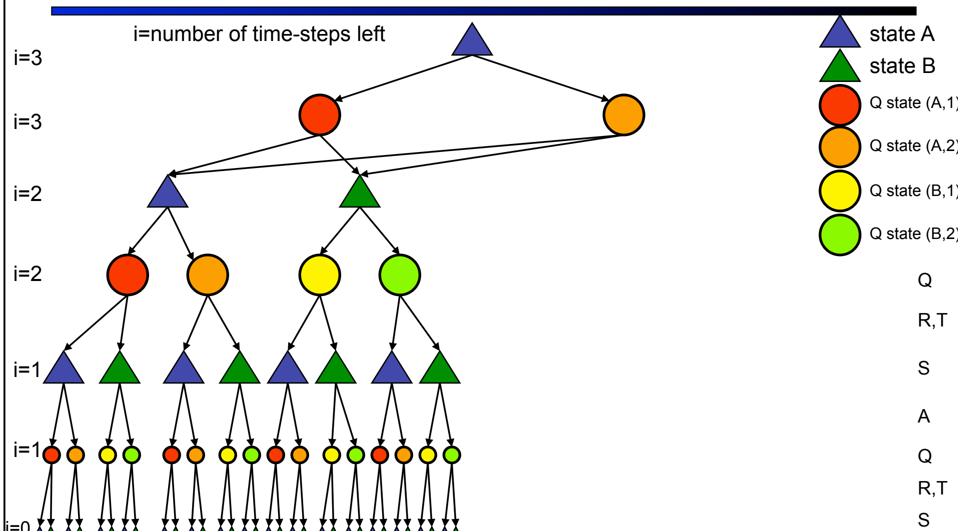


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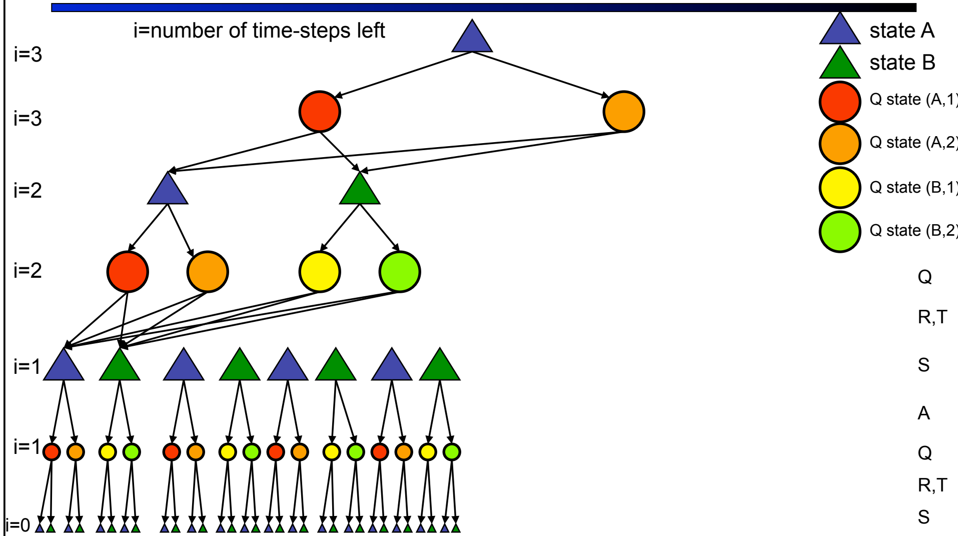


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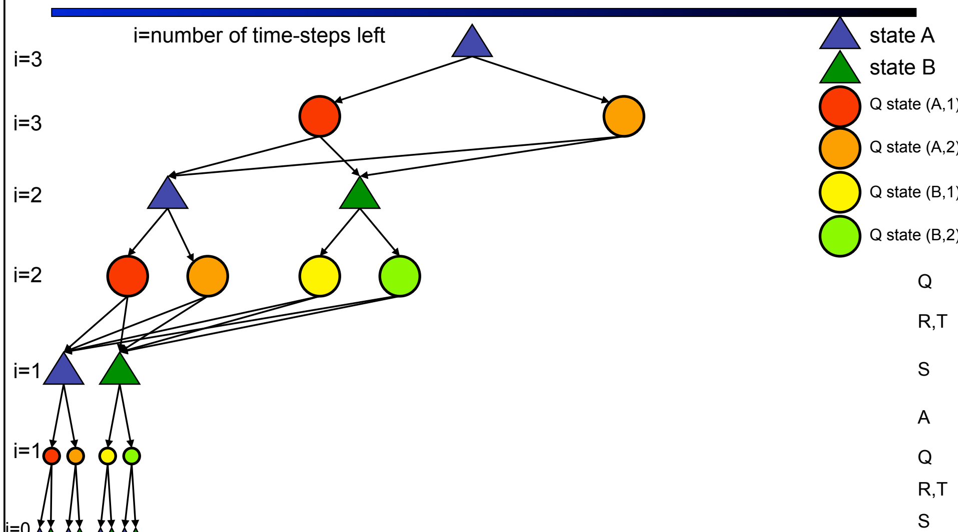


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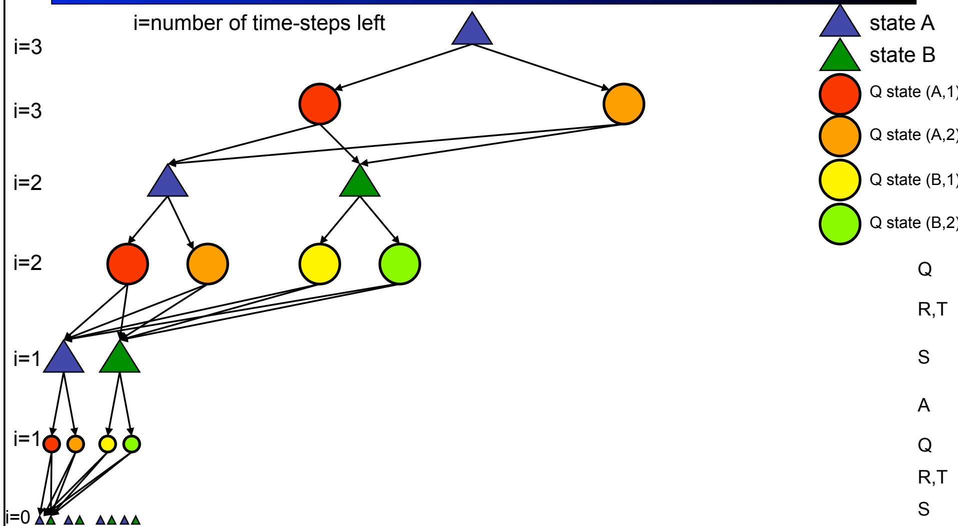


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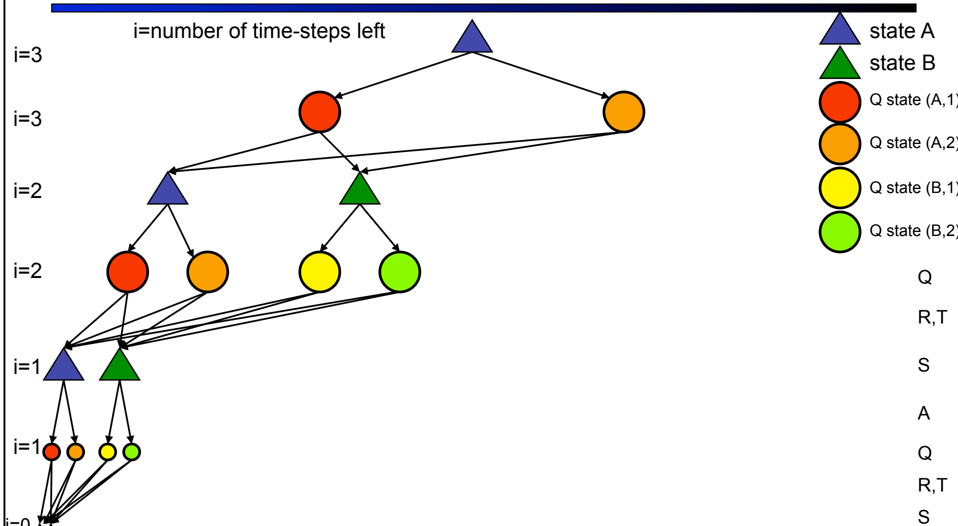
Example MDP used for illustration has two states, S = {A, B}, and two actions, A = {1, 2}

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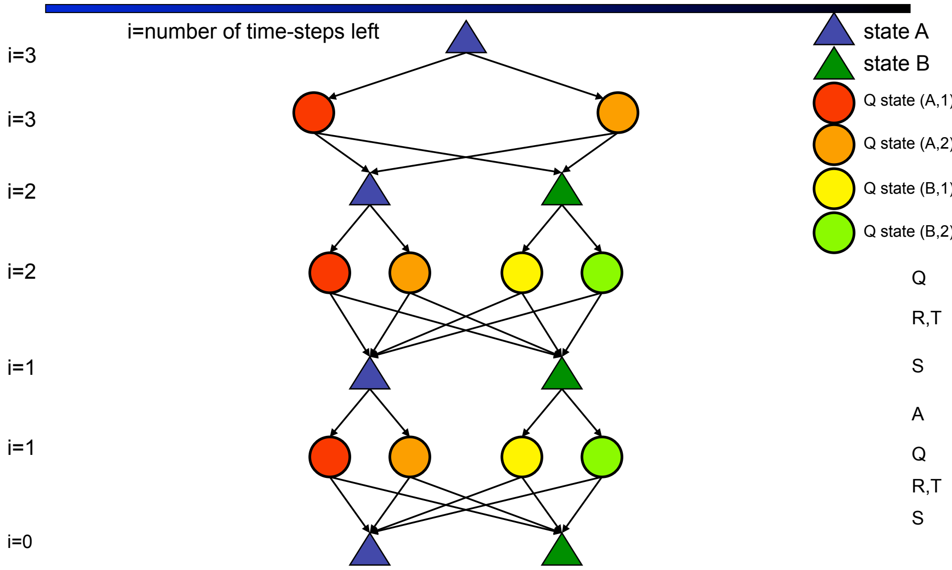


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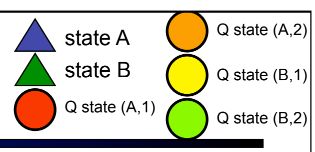
# Value Iteration Performs this Computation Bottom to Top

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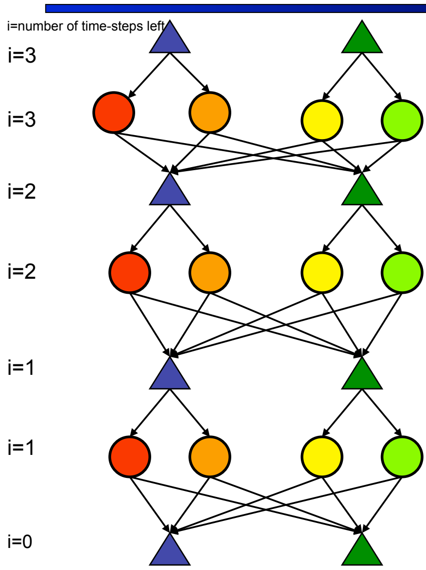


$$\forall s \in S, V_3^*(s) = \max_{a \in A} Q_3^*(s,a)$$

$$\forall s \in S, \forall a \in A$$

$$Q_3^*(s,a) = \sum_{s' \in S} T(s,a,s')[R(s,a,s') + V_2^*(s')]$$

$$\forall s \in S, V_2^*(s) = \max_{a \in A} Q_2^*(s,a)$$

$$\forall s \in S, \forall a \in A$$

$$Q_2^*(s,a) = \sum_{s' \in S} T(s,a,s')[R(s,a,s') + V_1^*(s')]$$

$$\forall s \in S, V_1^*(s) = \max_{a \in A} Q_1^*(s,a)$$

$$\forall s \in S, \forall a \in A$$

$$Q_1^*(s,a) = \sum_{s' \in S} T(s,a,s')[R(s,a,s') + V_0^*(s')]$$

Initialization: $V_0^*(A) = 0 \quad V_0^*(B) = 0$

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# Value Iteration for Finite Horizon H and no Discounting

- Initialization: $\forall s \in S : V_0^*(s) = 0$
- For i =1, 2, …, H
  - For all s $\in$ S
    - For all a $\in$ A: $Q_i^*(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + V_{i-1}^*(s')]$
    - $V_i^*(s) = \max_{a \in A} Q_i^*(s,a)$ $\quad \pi_i^*(s) = \arg\max_{a \in A} Q_i^*(s,a)$

- $V^*_i(s)$ : the expected sum of rewards accumulated when starting from state s and acting optimally for a horizon of i time steps.
- $Q^*_i(s)$: the expected sum of rewards accumulated when starting from state s with i time steps left, and when first taking action and acting optimally from then onwards
- How to act optimally? Follow optimal policy $\pi^*_i$(s) when i steps remain:

$$\pi_i^*(s) = \max_a Q_i^*(s,a) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + V_{i-1}^*(s')]$$

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# Value Iteration for Finite Horizon H and with Discounting

- Initialization: $\forall s \in S : V_0^*(s) = 0$
- For i =1, 2, …, H
  - For all s $\in$ S
    - For all a $\in$ A: $Q_i^*(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_{i-1}^*(s')]$
    - $V_i^*(s) = \max_{a \in A} Q_i^*(s,a)$ $\quad \pi_i^*(s) = \arg\max_{a \in A} Q_i^*(s,a)$

- $V^*_i(s)$ : the expected sum of *discounted* rewards accumulated when starting from state s and acting optimally for a horizon of i time steps.
- $Q^*_i(s)$: the expected sum of *discounted* rewards accumulated when starting from state s with i time steps left, and when first taking action and acting optimally from then onwards
- How to act optimally? Follow optimal policy $\pi^*_i$(s) when i steps remain:

$$\pi_i^*(s) = \arg\max_a Q_i^*(s,a) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_{i-1}^*(s')]$$

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# Value Iteration Rewritten

- Initialization: $\forall s \in S : V_0^*(s) = 0$
- For i =1, 2, …, H
  - For all s $\in$ S
    - For all a $\in$ A: $Q_i^*(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_{i-1}^*(s')]$
    - $V_i^*(s) = \max_{a \in A} Q_i^*(s,a)$

Maps more directly to how you would code value iteration

This is just substituting the expression for $Q^*_i$.

- Initialization: $\forall s \in S : V_0^*(s) = 0$
- For i =1, 2, …, H
  - For all s $\in$ S
    - $V_i^*(s) = \max_{a \in A} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_{i-1}^*(s')]$

Rewritten version is convenient for our ensuing discussion of convergence properties

Having done so, makes it very explicit that we can think of Value Iteration as computing the sequence $V_0$, $V_1$, $V_2$, …

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# Convergence

**Value Iteration**

- Initialization: $\forall s \in S : V_0^*(s) = 0$
- For i =1, 2, …, H
  - For all s $\in$ S
    - $V_i^*(s) = \max_{a \in A} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_{i-1}^*(s')]$

- Question we are about to answer is whether this procedure converges, i.e.,

  what happens for H -> $\infty$ ?

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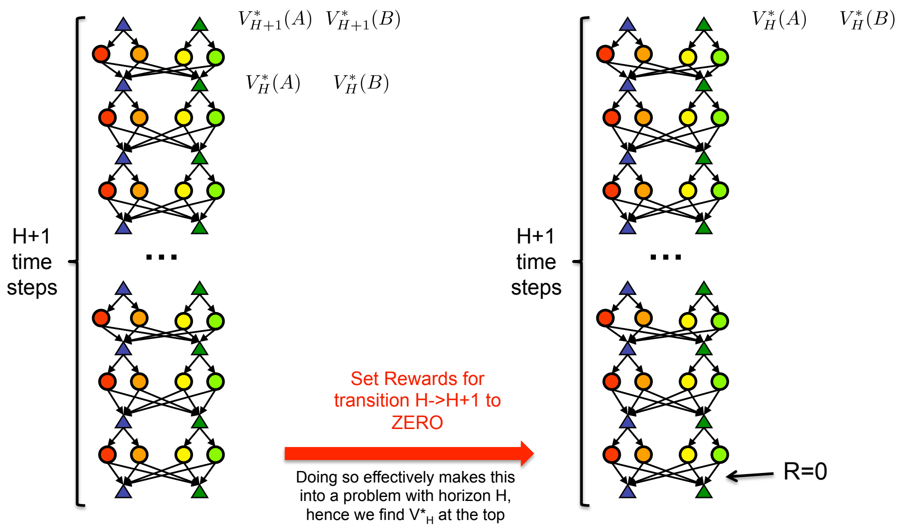
# Convergence

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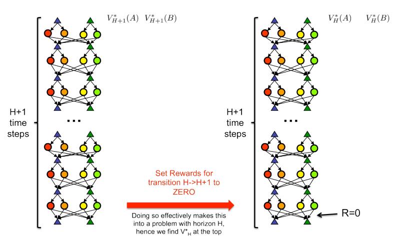
# Convergence



How different can $V^*_H$ and $V^*_{H+1}$ be?

- Both are the optimal expected sum of rewards when acting for H+1 time steps in the same MDP, except that for $V^*_H$ the rewards are set to zero for the transition H->H+1
- In the best possible scenario for $V^*_{H+1}$, one is able to achieve $V^*_H$ in the first H time steps, and then $\gamma^{H+1} \max_{s,a,s'} R(s,a,s')$ in the last time step
  [you can't do better than that, make sure you understand why]
- In the worst possible scenario for $V^*_{H+1}$, one is able to achieve $V^*_H$ in the first H time steps, and then $\gamma^{H+1} \min_{s,a,s'} R(s,a,s')$ in the last time step
  [you can't do worse than that, make sure you understand why

Hence we have: $$|V^*_H(s) - V^*_{H+1}(s)| \leq \gamma^{H+1} \max_{s,a,s'} |R(s,a,s')|$$

Hence the difference decays exponentially, and hence the series $V^*_1$, $V^*_2$, $V^*_3$, … converges to a limit, which we call V*.

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# Value Iteration Convergence

**Theorem.** Value iteration converges. At convergence, we have found the optimal value function V* for the discounted infinite horizon problem, which satisfies the Bellman equations

$$\forall s \in S: \quad V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V^*(s') \right]$$

- Now we know how to act for infinite horizon with discounted rewards!
  - Run value iteration till convergence.
  - This produces V*, which in turn tells us how to act, namely following:

$$\pi^*(s) = \arg\max_{a \in A} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- Note: the infinite horizon optimal policy is stationary, i.e., the optimal action at a state s is the same action at all times. (Efficient to store!)

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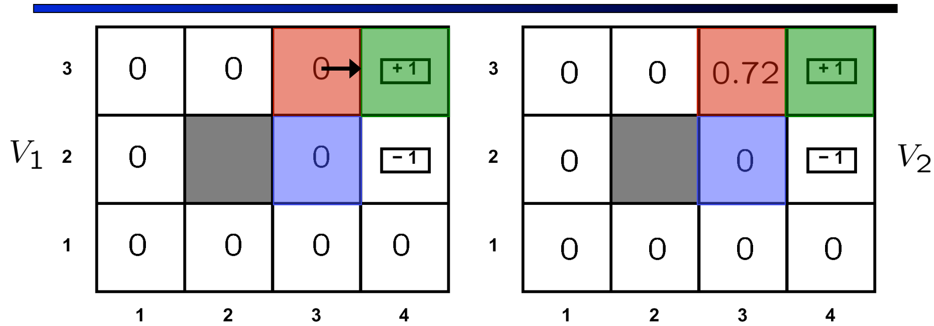
*Example: γ=0.9, living reward=0, noise=0.2*

# Example: Bellman Updates




$$V_{i+1}(s) = \max_a \sum_{s'} T(s,a,s') \left[ R(s,a,s') + \gamma V_i(s') \right]$$

$$V_2(\langle 3,3 \rangle) = \sum_{s'} T(\langle 3,3 \rangle, \text{right}, s') \left[ R(\langle 3,3 \rangle) + 0.9\, V_1(s') \right]$$

*max happens for a=right, other actions not shown*

$$= 0.9\, [0.8 \cdot 1 + 0.1 \cdot 0 + 0.1 \cdot 0]$$

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## Convergence (from Contraction Perspective)*

- Define the max-norm: $||U|| = \max_s |U(s)|$
- Theorem: For any two approximations U and V

$$||U_{i+1} - V_{i+1}|| \leq \gamma \, ||U_i - V_i||$$

  - I.e. any distinct approximations must get closer to each other, so, in particular, any approximation must get closer to the true U and value iteration converges to a unique, stable, optimal solution
- Theorem:

$$||U_{i+1} - U_i|| < \epsilon, \Rightarrow ||U_{i+1} - U|| < 2\epsilon\gamma/(1-\gamma)$$

  - I.e. once the change in our approximation is small, it must also be close to correct

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## Reminder: Computing Actions

- Which action should we chose from state s:
  - Given optimal values V*?

$$\arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

  - Given optimal q-values Q*?

$$\arg\max_a Q^*(s,a)$$

  - Lesson: actions are easier to select from Q’s!

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# Our Status

- Markov Decision Processes (MDPs)
  - ✓ Formalism
  - ✓ Value iteration
    - In essence a graph search version of expectimax, but
      - there are rewards in every step (rather than a utility just in the terminal node)
      - ran bottom-up (rather than recursively)
      - can handle infinite duration games
  - Policy Evaluation and Policy Iteration

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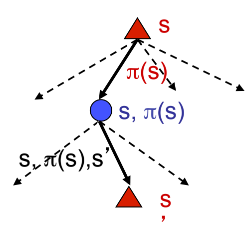
# Policy Evaluation

- Another basic operation: compute the utility of a state s under a fix (general non-optimal) policy




- Define the utility of a state s, under a fixed policy $\pi$:

  $V^{\pi}(s)$ = expected total discounted rewards (return) starting in s and following $\pi$

- Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

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## Policy Evaluation

- How do we calculate the V' s for a fixed policy?
- Idea one: modify Bellman updates

$$V_0^\pi(s) = 0$$

$$V_{i+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_i^\pi(s')]$$

- Idea two: it' s just a linear system, solve with Matlab (or whatever)

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## Policy Iteration

- Alternative approach:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges
- This is policy iteration
  - It' s still optimal!
  - Can converge faster under some conditions

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# Policy Iteration

- Policy evaluation: with fixed current policy $\pi$, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$



# Policy Iteration Guarantees

Policy Iteration iterates over:


- Policy evaluation: with fixed current policy $\pi$, find values with simplified Bellman updates:
  - Iterate until values converge

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} T(s, \pi_k(s), s') \left[R(s, \pi_k(s), s') + \gamma V_i^{\pi_k}(s')\right]$$

- Policy improvement: with fixed utilities, find the best action according to one-step look-ahead

$$\pi_{k+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_k}(s')\right]$$


**Theorem.** Policy iteration is guaranteed to converge and at convergence, the current policy and its value function are the optimal policy and the optimal value function!

Proof sketch:

(1) *Guarantee to converge*: we will not prove this, but the proof proceeds by first showing that in every step the policy improves. This means that a given policy can be encountered at most once. This means that after we have iterated as many times as there are different policies, i.e., (number actions)$^{\text{(number states)}}$, we must be done and hence have converged.

(2) *Optimal at convergence*: by definition of convergence, at convergence $\pi_{k+1}$(s) = $\pi_k$(s) for all states s. This means $\forall s \;\; V^{\pi_k}(s) = \max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V_i^{\pi_k}(s')\right]$

Hence $V^{\pi_k}$ satisfies the Bellman equation, which means $V^{\pi_k}$ is equal to the optimal value function V*.

# Comparison

- In value iteration:
  - Every pass (or "backup") updates both utilities (explicitly, based on current utilities) and policy (possibly implicitly, based on current policy)
- In policy iteration:
  - Several passes to update utilities with frozen policy
  - Occasional passes to update policies
- Hybrid approaches (asynchronous policy iteration):
  - Any sequences of partial updates to either policy entries or utilities will converge if every state is visited infinitely often

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# Asynchronous Value Iteration*

- In value iteration, we update every state in each iteration
- Actually, *any* sequences of Bellman updates will converge if every state is visited infinitely often
- In fact, we can update the policy as seldom or often as we like, and we will still converge
- Idea: Update states whose value we expect to change:
  If $|V_{i+1}(s) - V_i(s)|$ is large then update predecessors of s

# MDPs recap

- Markov decision processes:
  - States S
  - Actions A
  - Transitions P(s'|s,a) (or T(s,a,s'))
  - Rewards R(s,a,s') (and discount $\gamma$)
  - Start state $s_0$
- Solution methods:
  - Value iteration (VI)
  - Policy iteration (PI)
  - Asynchronous value iteration*
- Current limitations:
  - Relatively small state spaces
  - Assumes T and R are known

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